Factoring Polynomials

Sept 28/2016

Remainder Theorem: When a polynomial, $f(x)$, is divided by $(x-a)$, the remainder is equal to $f(a)$.

Factor Theorem:

If the remainder, or $f(a)$, is equal to zero, then $x-a$ is a factor of the polynomial $f(x)$

For example, recall:

$$\frac{3x^3-5x^2-7x-1}{x-3}=3x^2+4x+5+\frac{14}{x-3}$$

Ex.1 Use the factor theorem to determine one factor of

$$f(x) = x^3 + 4x^2 + x - 6$$

then completely factor the function.

$f(0) = -6$ $\quad$ x is NOT a factor

$$f(1) = (1)^3 + 4(1)^2 + (1) - 6$$

$$= 0$$

$(x-1)$ is a factor

$$\begin{array}{r l}
 & x^2+5x+6 \\ \hline
x-1 \,\big) & x^3+4x^2+x-6 \\
 & x^3-x^2 \quad \downarrow \quad \downarrow \\ \hline
 & \quad\; 5x^2+x \\
 & \quad\; 5x^2-5x \\ \hline
 & \qquad\quad 6x-6 \\
 & \qquad\quad 6x-6 \\ \hline
 & \qquad\qquad 0
\end{array}$$

$$x^3+4x^2+x-6 = (x-1)(x^2+5x+6)$$

$$= (x-1)(x+2)(x+3)$$

A rational number can be expressed as a fraction with an integer numerator and denominator (but no division by zero).

$$\frac{a}{b} \quad \text{where} \quad a,b \in \mathbb{Z}, b \neq 0$$

A rational root is a zero which is a rational number. For a polynomial, roots can be expressed as factors:

$$\left(x - \frac{a}{b}\right)$$, or, more commonly, $$(bx - a)$$

$$3\left(x - \frac{5}{3}\right)$$

$$3x - 5$$

The <u>rational roots test</u> allows us to limit our search for roots using the leading term and the constant (last) term.

For a polynomial in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

the <u>possible</u> rational roots are

$$\frac{\text{all factors of constant term}}{\text{all factors of leading coefficient}}$$

note: Some roots are <u>irrational</u>, and there is no guarantee that the rational root test will be successful.

Therefore, not all polynomials will be factorable.

For example, $y = 3x^2 + 10x - 8$ has a constant term -8 and a leading coefficient of 3.

factors of 8 are 1, 2, 4, 8

factors of 3 are 1, 3

possible rational roots are $\dfrac{\pm 1, 2, 4, 8}{1, 3}$

As a list: $\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$

Using the factor theorem, we can test each one of these until $f(a) = 0$ For this quadratic, $f(-4) = 0$, $f\left(\frac{2}{3}\right) = 0$

$$\begin{aligned} 3x^2 + 10x - 8 &= (x+4)(3)\left(x - \tfrac{2}{3}\right) \\ &= (x+4)(3x-2) \end{aligned}$$

$$2x + 5 = 2\left(x + \frac{5}{2}\right)$$

Ex.2 Determine all possible rational roots for

$$f(x) = 2x^3 + 7x^2 - 64x - 105$$

then show that $2x + 3$ is a factor.

$105 = 1 \times 105 = 3 \times 35 = 5 \times 21 = 7 \times 15$

$2 = 1 \times 2$

possible roots : $\dfrac{\pm 1, 3, 5, 7, 15, 21, 35, 105}{1, 2}$

$\pm 1, 3, 5, 7, 15, 21, 35, 105, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{15}{2}, \frac{21}{2}, \frac{35}{2}, \frac{105}{2}$

$2x + 3 \longrightarrow$ root at $-\frac{3}{2}$

$f\left(-\frac{3}{2}\right) =$

Assigned Work:

p.176 # [1 - 3][basics],
4bf, 5ac, 6abc, 7df, 8, 9, 10, 13, 17
f

6.(a) $f(x) = x^3 - 3x^2 - 10x + 24$

$\pm\ 1, 2, 3, 4, 6, 8, 12, 24$

$$f(2) = 8 - 3(4) - 10(2) + 24$$
$$= 8 - 12 - 20 + 24$$
$$= 0$$

$\therefore x - 2$ is a factor

$$\begin{array}{r} x^2 - x - 12 \\ x-2\,\overline{)\,x^3 - 3x^2 - 10x + 24} \end{array}$$

$(x-4)(x+3)$

$f(x) = (x-2)(x-4)(x+3)$

(d) $f(x) = x^4 + \ldots\ldots + 64$

$$\pm \frac{1, 2, 4, 8, 16, 32, 64}{1}$$

$f(-1) = 0 \implies x+1$ is a factor

$$\begin{array}{r l} & x^3+2x^2-40x+64 \\ x+1 \big) & \overline{\quad f(x) \qquad\qquad} \end{array}$$

$$\downarrow$$

$$g(x)$$

$g(2) = 0 \quad x-2$ is a factor

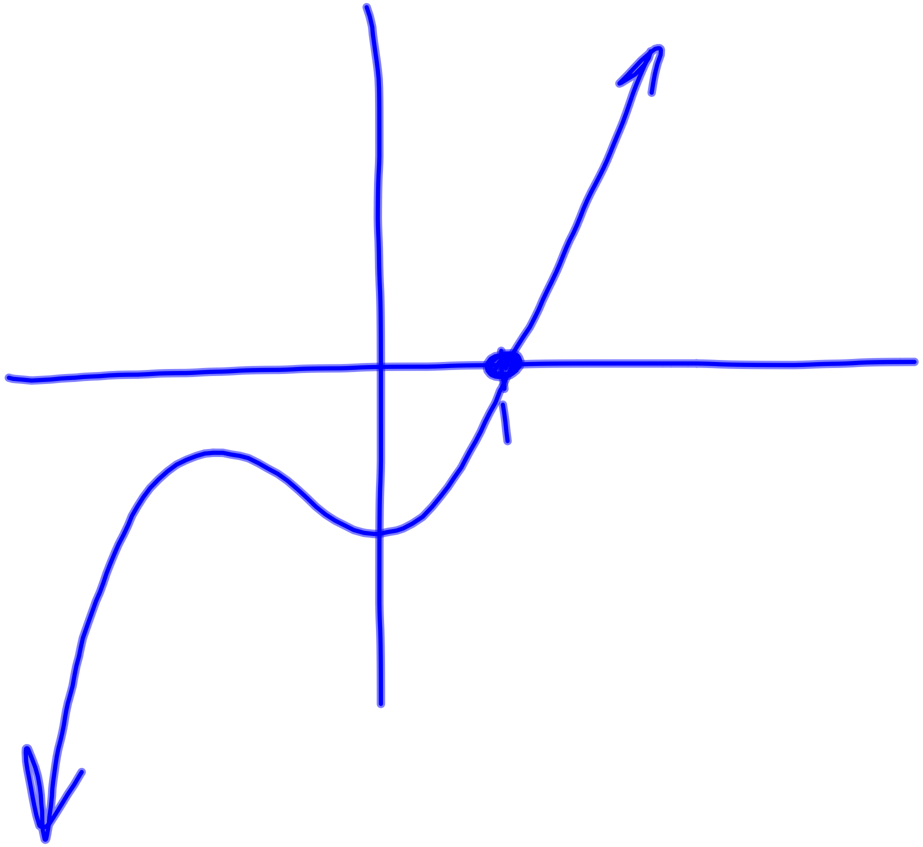
$$\begin{array}{r l} & x^2+4x-32 \\ x-2 \big) & \overline{\quad g(x) \qquad\quad} \end{array}$$

$$\downarrow$$

$$(x+8)(x-4)$$

$$f(x) = (x+1)(x-2)(x+8)(x-4)$$

7(f) $f(x) = (x^2+1)^2(x-1)$



$x^2+1=0$

$x^2=-1$

9. $f(x) = 12x^3 + kx^2 - x - 6$

$2x-1$ is a factor

$0 = 12\left(\frac{1}{2}\right)^3 + \underline{k}\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6$

?

$\vdots$

$k =$ _____

$f\left(\frac{1}{2}\right) = 0$

10. $f(x) = ax^3 - x^2 + 2x + b$

$f(1) = 10$

$f(2) = 51$

13. $f(x)$

$f(-2) = R_1$ $\qquad$ $f(2) = R_2$

$$R_1 = 2R_2$$

$$f(-2) = 2f(2)$$

$$\boxed{\qquad\qquad k \qquad\qquad} = 2\boxed{\qquad\qquad k \qquad\qquad}$$

$\vdots$

$k =$ ________

17 prove $x+a$ is a factor of

$$f(x)=(x+a)^5+(x+c)^5+(a-c)^5$$

evaluate $f(-a) =$

$$\vdots$$

is it zero?