Double Angle Formulas

Nov.21/2016

These can be derived from the compound angle formulas for sine, cosine, and tangent.

recall: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$\sin(2\theta) = \sin(\theta+\theta)$$

$$= \underbrace{\sin\theta\cos\theta} + \underbrace{\cos\theta\sin\theta}$$

$$= 2\sin\theta\cos\theta$$

recall: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos(2\theta) = \cos(\theta+\theta)$$

$$= \cos\theta \cos\theta - \sin\theta \sin\theta$$

$$= (\cos\theta)^2 - (\sin\theta)^2$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$ ✓

$$= (1-\sin^2\theta) - \sin^2\theta$$

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta}$$

$$= \cos^2\theta - (1-\cos^2\theta)$$

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$$

Summary:

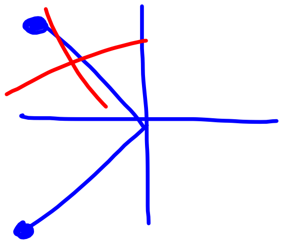
$$\sin(2\theta) = 2\ \sin\theta\ \cos\theta$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\ \cos^2\theta - 1 \\ &= 1 - 2\ \sin^2\theta\end{aligned}$$

$$\tan(2\theta) = \frac{2\ \tan\theta}{1 - \tan^2\theta}$$

Ex.1 If $\cos\theta = \frac{-2}{3}$ and $\pi \le \theta < 2\pi$, determine the values of $\sin 2\theta$ and $\cos 2\theta$.

Q3, Q4



$\cos\theta = \frac{x}{r}$ $x = -2$, $r = 3$

$(-2)^2 + y^2 = 3^2$

$y = \pm\sqrt{5}$

$\cos\theta = \frac{-2}{3} \rightarrow$ Q2, Q3

$y = -\sqrt{5}$, Q3

$\pi \le \theta < 2\pi \rightarrow$ Q3, Q4

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$

$= 2\left(\frac{-\sqrt{5}}{3}\right)\left(\frac{-2}{3}\right)$

$\sin 2\theta = \frac{4\sqrt{5}}{9}$

$\cos 2\theta = 2\cos^2\theta - 1$

$= 2\left(\frac{-2}{3}\right)^2 - 1$

$= 2\left(\frac{4}{9}\right) - 1$

$= \frac{8}{9} - \frac{9}{9}$

$\cos 2\theta = \frac{-1}{9}$

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Assigned Work:

Derive tan(2x)
p.407 # 1-3, 5, 6, 8, 9 (see p.406), 13

3b, 1e, 2f any

$\sin(4\theta)$ let $\alpha = 2\theta$

$= \sin 2\alpha$

$= 2 \sin \alpha \cos \alpha$

$= 2 \sin 2\theta \cos 2\theta$

1(e) $4\sin\theta\cos\theta$

$= 4\left(\frac{\sin 2\theta}{2}\right)$

$= 2\sin 2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{\sin 2\theta}{2} = \sin\theta\cos\theta$

$= 2(2\sin\theta\cos\theta)$

$= 2\sin 2\theta$

2(f) $2\tan 60° \cos^2 60°$

$= 2\,\frac{\sin 60°}{\cancel{\cos 60°}}\cos^{\cancel{2}} 60°$ $\qquad \frac{xy^2}{y}$

$= 2\sin 60° \cos 60°$ $\qquad = xy$

$= \sin 120°$

3(b) $\cos 3x$ $\qquad\qquad$ $\cos 2\theta$

$= \cos[2(1.5x)]$ $\qquad$ $= 2\cos^2\theta - 1$

$= 2\cos^2(1.5x) - 1$

OR

$= \cos^2(1.5x) - \sin^2(1.5x)$

OR

$= 1 - 2\sin^2(1.5x)$

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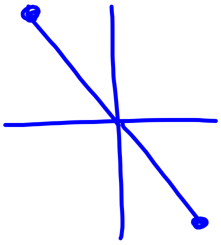
5. $\tan\theta = \frac{-7}{24}$ $\frac{y}{x}$ $\frac{\pi}{2} < \theta < \pi$

Q2, Q4 Q2



$x = -24$

$y = 7$

$r = \sqrt{(7^2)+(-24)^2}$

$= 25$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{7}{25}\right)\left(\frac{-24}{25}\right)$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{\cancel{2}^{1}\left(\frac{-7}{\cancel{24}_{12}}\right)}{1-\left(\frac{-7}{24}\right)^2}$$
$$= \frac{\frac{-7}{12}}{1-\frac{49}{576}}$$
$$= \frac{\frac{-7}{12}}{\frac{576-49}{576}}$$
$$= \frac{-7}{12} \times \frac{576}{527}$$
$$= \frac{-336}{527}$$

8. $2\tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$

$$2a = 1 - \underline{\tan 2x}\tan^2 x - 2\tan x + \tan 2x$$

$$2a = 1 - \frac{2\tan x}{(1-\tan^2 x)}\tan^2 x - 2\tan x + \frac{2\tan x}{1-\tan^2 x}$$

$$2a = \frac{1-\tan^2 x - 2\tan^3 x - 2\tan x(1-\tan^2 x) + 2\tan x}{1-\tan^2 x}$$

$$= \frac{1-\tan^2 x \cancel{-2\tan^3 x} \cancel{-2\tan x} \cancel{+2\tan^3 x} \cancel{+2\tan x}}{1-\tan^2 x}$$

$$= \frac{1-\tan^2 x}{1-\tan^2 x}$$

$$2a = 1$$

$$a = \frac{1}{2}$$

9. $\sin\left(\frac{\pi}{8}\right) = ?$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\frac{\pi}{4} = 2\left(\frac{\pi}{8}\right)$$

$\cos 2\theta$
$= 1 - 2\sin^2\theta$

$$\cos\frac{\pi}{4} = \cos\left(2\left(\frac{\pi}{8}\right)\right)$$

$$\frac{1}{\sqrt{2}} = 1 - 2\sin^2\frac{\pi}{8}$$

$$\frac{1}{\sqrt{2}} = 1 - 2x^2 \qquad x = \sin\frac{\pi}{8}$$

$$2\sin^2\frac{\pi}{8} = 1 - \frac{1}{\sqrt{2}}$$

$$2\sin^2\frac{\pi}{8} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\sin^2\frac{\pi}{8} = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\sin\frac{\pi}{8} = \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}} \qquad \frac{\pi}{8} \text{ is acute}$$

$$\sin\frac{\pi}{8} = \sqrt{\frac{\sqrt{4}-\sqrt{2}}{2\sqrt{4}}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

13.(c) $\cos\frac{x}{2} = ?$ $\sin^2 x = \frac{8}{9}$ $\underbrace{\frac{\pi}{2} \le x \le \pi}_{Q2}$

① need to relate $\frac{x}{2}$ to x to use double-angle identities.

$$x = \underbrace{2\left(\frac{x}{2}\right)}$$

2θ, so $\theta = \frac{x}{2}$ in our formulas.

② which identity is useful?

$\sin 2\theta = 2\sin\theta\cos\theta$ replace $\theta = \frac{x}{2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

given ✓ ??? wanted ✓ $\therefore$ no good

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ → dead ends!

$\cos x = 2\cos^2\frac{x}{2} - 1$

$\cos x = 1 - 2\sin^2\frac{x}{2}$ → dead ends!

We know $\sin^2 x = \frac{8}{9}$

$$1 - \cos^2 x = \frac{8}{9}$$
$$1 - \frac{8}{9} = \cos^2 x$$
$$\frac{1}{9} = \cos^2 x$$

$\cos x = \pm\frac{1}{3}$, but $x \in Q2$

$$\therefore \cos x = -\frac{1}{3}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$
$$-\frac{1}{3} = 2\cos^2\frac{x}{2} - 1$$
$$\frac{2}{3} = 2\cos^2\frac{x}{2}$$
$$\frac{1}{3} = \cos^2\frac{x}{2}$$
$$\pm\sqrt{\frac{1}{3}} = \cos\frac{x}{2}$$

but $\frac{\pi}{2} \le x \le \pi$

$\Rightarrow \frac{\pi}{4} \le \frac{x}{2} \le \frac{\pi}{2}$

$\Rightarrow \frac{x}{2}$ in Q1

$$\cos\frac{x}{2} = \frac{1}{\sqrt{3}}$$
$$\cos\frac{x}{2} = \frac{\sqrt{3}}{3}$$

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