Solving Systems of Equations using Matrices

Consider a system of three equations in three unknowns:

$$x + 2y + 2z = 9 \quad 1$$
$$x + y = 1 \quad 2$$
$$2x + 3y - z = 1 \quad 3$$

A matrix is an equivalent representation of this information:

$$\begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{cccc} x & y & z & \text{const} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right] \end{array}$$

coefficient matrix

augmented matrix
(includes constants)

Elementary row operations can be used to modify and solve systems of equations in matrix form.

(1) Multiply a row by a nonzero constant.
(2) Interchange any pair of rows.
(3) Add (or subtract) a multiple of one row to a second row, replacing the second row.

replaced

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right] \xrightarrow{-r_1} \left[\begin{array}{ccc|c} -1 & -2 & -2 & -9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right] \xrightarrow[r_2]{r_1 + r_2} \left[\begin{array}{ccc|c} -1 & -2 & -2 & -9 \\ 0 & -1 & -2 & -8 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

-1 (row1) + row2 replaces row2

To find solutions to the system, we continue to manipulate the matrix until it is in row-echelon form:

$$\left[\begin{array}{ccc|c} a & b & c & p \\ 0 & d & e & q \\ 0 & 0 & f & r \end{array}\right]$$

notice the upper-triangle of non-zero values, and the lower-triangle of zeroes

Continuing with our previous example, row 1 and row 2 are already in row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 2 & 3 & -1 & 1 \end{array}\right] \xrightarrow{-2r_1+r_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & -1 & -5 & -17 \end{array}\right]$$

0 0

$$\xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

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The row-echelon augmented matrix is equivalent to a system of new, but equivalent, equations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right] \Leftrightarrow \begin{aligned} x+2y+2z &= 9 \\ -y-2z &= -8 \\ -3z &= -9 \end{aligned}$$

This system is easily solved for z, and the rest can be solved by back-substitution. This technique is called Gaussian Elimination.

$-3z = -9$

$\boxed{z = 3}$

$-y - 2z = -8$

$-y - 2(3) = -8$

$8 - 6 = y$

$\boxed{y = 2}$

$x + 2y + 2z = 9$

$x + 2(2) + 2(3) = 9$

$x + 4 + 6 = 9$

$\boxed{x = -1}$

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It is also possible to find solutions directly from the matrix by continuing to work towards a reduced row-echelon form. Solving this way is called Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \quad \text{which yields} \quad \begin{aligned} x &= a \\ y &= b \\ z &= c \end{aligned}$$

Ex.1 Write in reduced row-echelon form and solve

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

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(continued...)

Ex.1 Write in reduced row-echelon form and solve

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow{-r_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$$\xrightarrow{-\frac{r_3}{3}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2r_3+r_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2r_2+r_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2r_3+r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = -1$

$y = 2$

$z = 3$

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Assigned Work:

p.588 # 4, 6b, 8b, 10ace
p.594 # 1c, 3ace

read (diagrams & text) on
p.520-521
p.526-527

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