

Vertical & Horizontal Asymptotes

A. Vertical Asymptotes

The graph of $f(x)$ has a vertical asymptote, $x = c$ if one of the following is true:

$$\lim_{x \rightarrow c^-} f(x) = \pm\infty$$

(approach VA from left)

$$\lim_{x \rightarrow c^+} f(x) = \pm\infty$$

(approach VA from right)

Trivial Case:

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$

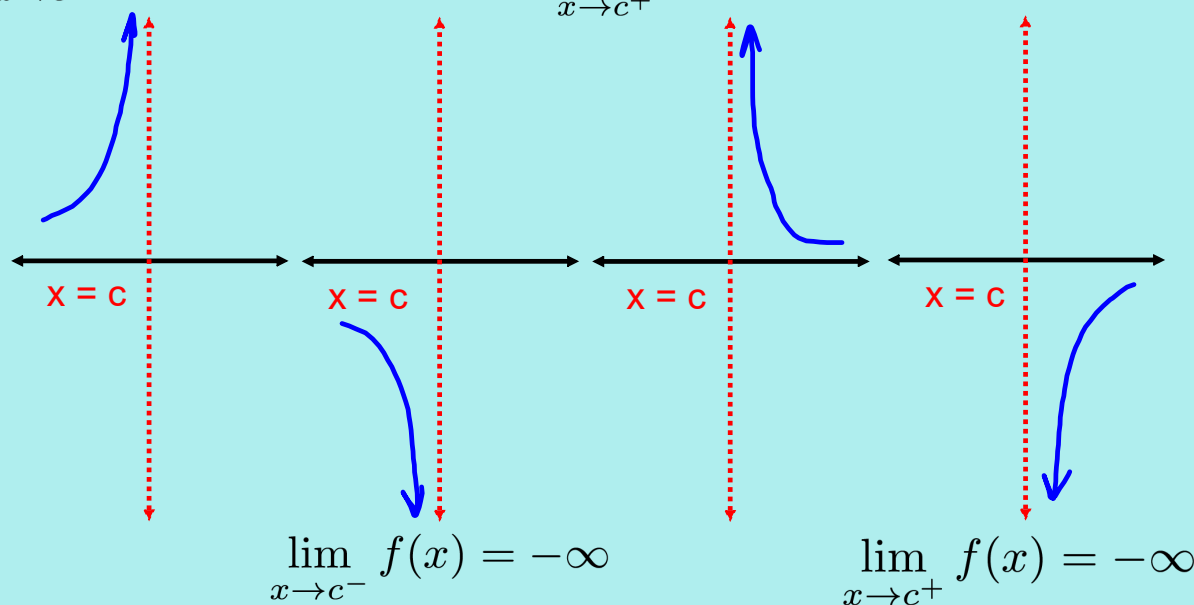
has a vertical asymptote at $x = c$ if

$$q(c) = 0 \text{ and } p(c) \neq 0.$$

Behaviour of Graph at Vertical Asymptotes

$$\lim_{x \rightarrow c^-} f(x) = +\infty$$

$$\lim_{x \rightarrow c^+} f(x) = +\infty$$



Ex.1 Determine any vertical asymptotes (VA) of

$$f(x) = \frac{x}{x^2 + x - 2}$$

and describe the behaviour of the graph for values near the asymptotes.

$$f(x) = \frac{x}{(x+2)(x-1)} \quad \underbrace{x \neq -2, x \neq 1}_{\text{VA's}}$$

VA: $x = -2$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{\overset{(-)}{x} \overset{-2}{\leftarrow}}{\underset{\substack{\uparrow \text{really} \\ \text{small} \\ \text{- number}}}{(-)(-)} \underset{-3}{(x+2)(x-1)}} \quad \lim_{x \rightarrow -2^+} \frac{\overset{(-)}{x}}{\underset{\substack{\text{(+small)} \text{ (-)} \text{ (-)}}}{(x+2)(x-1)}} = +\infty$$

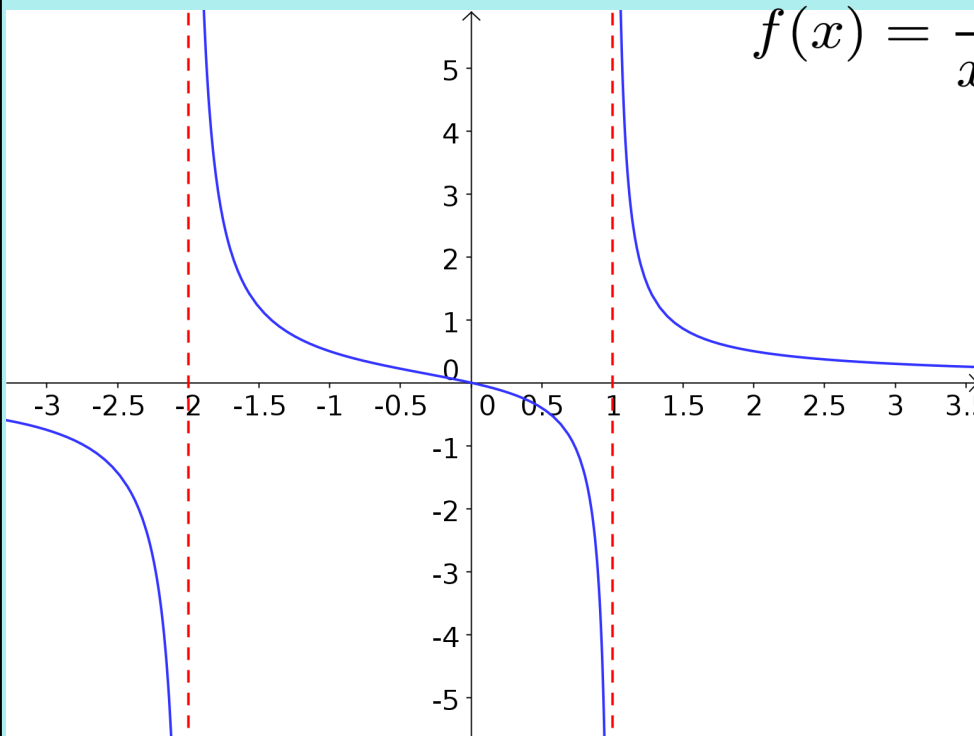
$$\lim_{x \rightarrow -2^-} f(x) = -\infty$$

VA: $x = 1$

$$\lim_{x \rightarrow 1^-} \frac{\overset{(+)}{x}}{\underset{\substack{\text{(+)} \text{ (-small)}}}{(x+2)(x-1)}} = -\infty \quad \lim_{x \rightarrow 1^+} \frac{x}{(x+2)(x-1)} = +\infty$$

Ex.1 Determine any vertical asymptotes (VA) of

$$f(x) = \frac{x}{x^2 + x - 2}$$



B. Horizontal Asymptotes

Consider the behaviour of the function as x tends to positive and negative infinity.

$$\text{If } \lim_{x \rightarrow +\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L ,$$

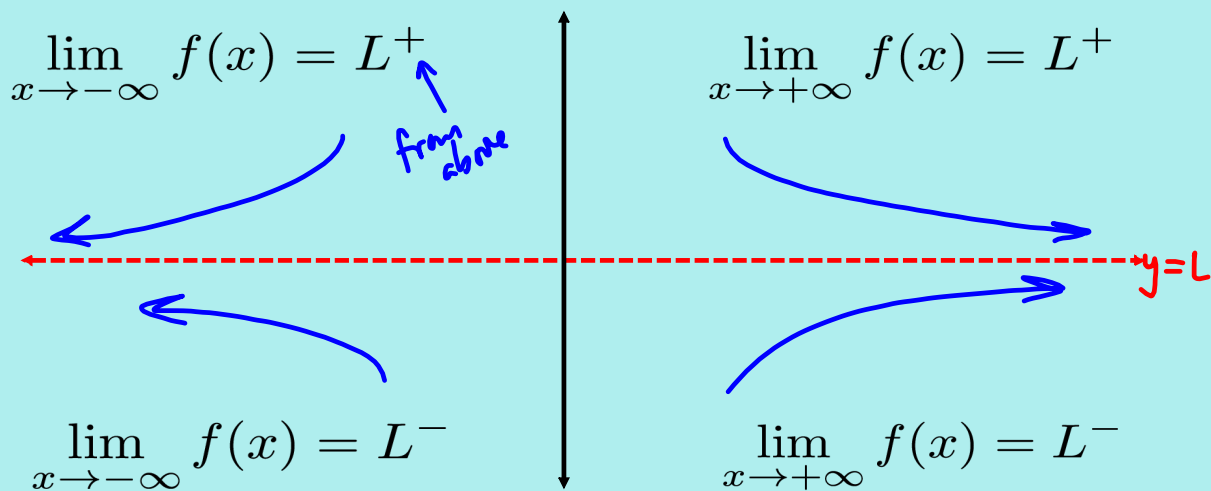
we say the line $y = L$ is a horizontal asymptote (HA).

Strategy for Rational Functions:

Factor the highest-order variable from the numerator and denominator (separately), and then apply the limit.

Note: It is important to consider whether the function approaches L from above or below.

Behaviour of Graph at Horizontal Asymptotes



Ex.2 Determine the equation of any horizontal asymptotes for

$$f(x) = \frac{3x^2}{x^2 - x - 6}$$

and behaviour.

factor x^2
out of
both num
& den.

$$f(x) = \frac{\cancel{x^2}(3)}{\cancel{x^2}\left(1 - \frac{x}{x^2} - \frac{6}{x^2}\right)}$$

$$= \frac{3}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

$$\text{HA: } \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{3}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

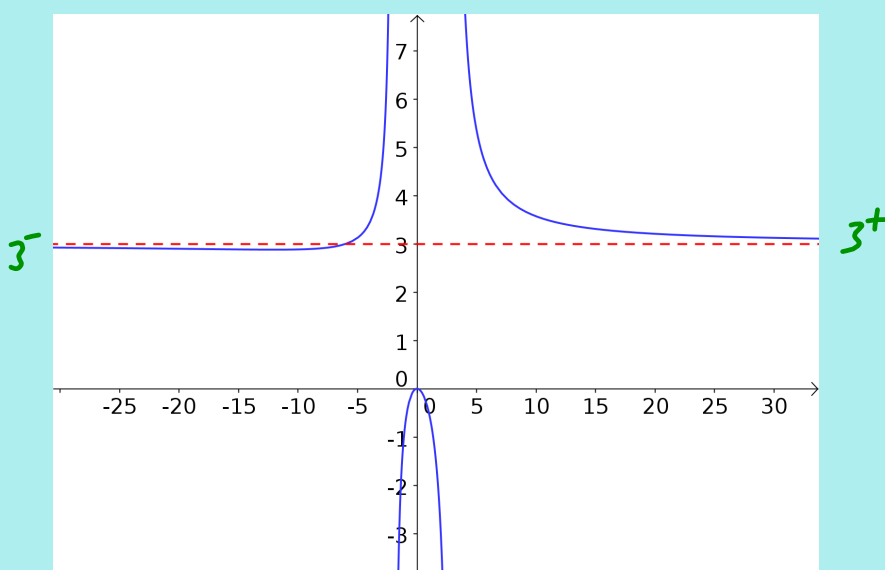
$$= 3^+$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{3}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

$$= 3^-$$

Ex.2 Determine the equation of any horizontal asymptotes for

$$f(x) = \frac{3x^2}{x^2 - x - 6}$$



Assigned Work:

p.193 # (3, 4, 5)(skip d),
6abd, 9abc, 11, 12, 13, 15*