<u>Concavity & Points of Inflection</u>

march 1/2018

Recall: $f'(x)$ describes the iRoC, or slope, of $f(x)$

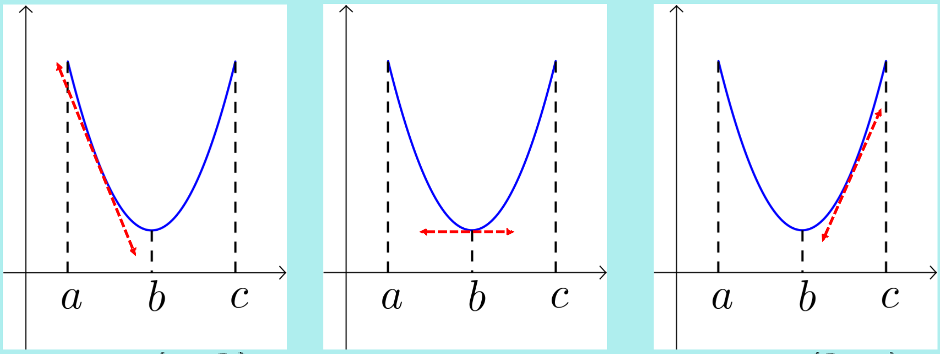
Similarly, $f''(x)$ describes the iRoC of $f'(x)$

The 1st derivative, or slope, tells us how the original function is changing (increasing or decreasing). If the slope is zero, the function may be at a local extrema.

The 2nd derivative tells us how the slope is changing, but we are generally more interested in the original function. Use f''(x) to determine the <u>concavity</u>, or curvature, of the function.

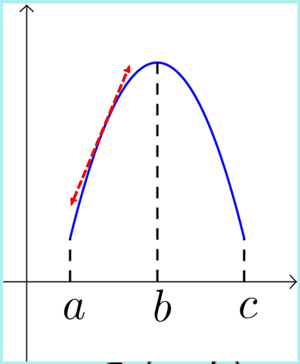
see Geogebra demo

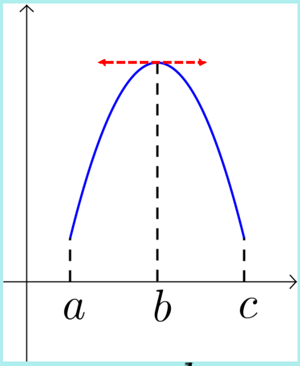
Ex. Identify key properties of each function (graph) in terms of the 1st and 2nd derivatives.

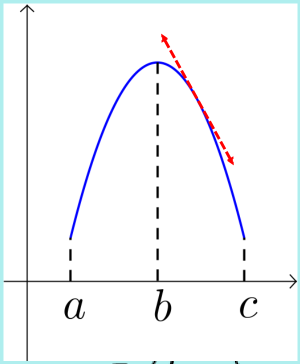



| | $x \in (a, b)$ | $x = b$ | $x \in (b, c)$ |
|---|---|---|---|
| $f(x)$ | decreasing | minimum | increasing |
| $f'(x)$ | negative | zero (critical value) | positive |
| $f''(x)$ | positive | positive | positive |

Ex. Identify key properties of each function (graph) in terms of the 1st and 2nd derivatives.

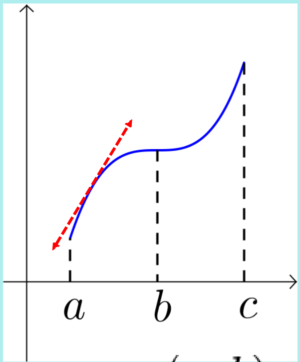


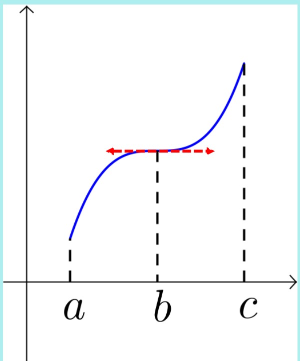


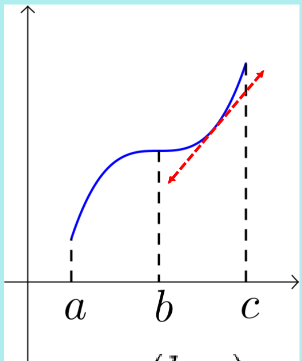



| | $x \in (a, b)$ | $x = b$ | $x \in (b, c)$ |
|---|---|---|---|
| $f(x)$ | increasing | maximum | decreasing |
| $f'(x)$ | positive | zero (critical value) | negative |
| $f''(x)$ | negative ⟶ | | |

Ex. Identify key properties of each function (graph) in terms of the 1st and 2nd derivatives.

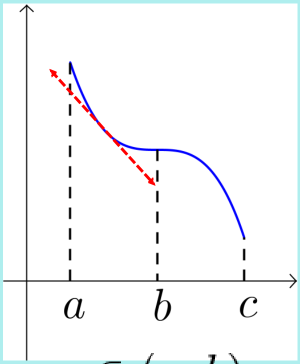


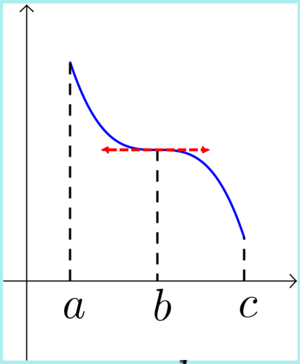


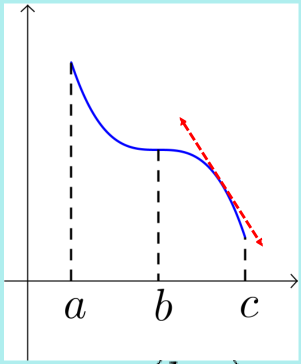



| | $x \in (a, b)$ | $x = b$ | $x \in (b, c)$ |
|---|---|---|---|
| $f(x)$ | increasing | point of inflection | increasing |
| $f'(x)$ | positive | positive (or zero) | positive |
| $f''(x)$ | negative | zero | positive |

Ex. Identify key properties of each function (graph) in terms of the 1st and 2nd derivatives.




| | $x \in (a,b)$ | $x = b$ | $x \in (b,c)$ |
|---|---|---|---|
| $f(x)$ | decreasing | point of inflection | decreasing |
| $f'(x)$ | negative | negative (or zero) | negative |
| $f''(x)$ | positive | zero | negative |

<table>
<tr><th></th><th colspan="2">detect critical values</th><th colspan="2">classify critical values</th></tr>
<tr><th></th><th>zero</th><th>undefined</th><th>positive</th><th>negative</th></tr>
<tr><td>$f(x)$</td><td>x-intercepts</td><td>VA or hole</td><td colspan="2"></td></tr>
<tr><td>$f'(x)$</td><td rowspan="2">critical value (extrema or POI)</td><td rowspan="2">if f(a) exists: critical value (extrema or POI)</td><td>increasing</td><td>decreasing</td></tr>
<tr><td>$f''(x)$</td><td>concave upward</td><td>concave downward</td></tr>
</table>

1st derivative test: Classify a critical value using increase or decrease on either side of x=a.

2nd derivative test: Classify using concavity at x=a (or on either side for f''(a) undefined).

Ex. Find and classify the critical values of the function

$$f(x) = 3x^5 - 25x^3 + 60x$$

Note: Polynomials have no asymptotes, so no limits required.

$$f'(x) = 15x^4 - 75x^2 + 60$$
$$= 15(x^4 - 5x^2 + 4)$$
$$= 15(x^2 - 4)(x^2 - 1)$$

set $f'(x) = 0$

$$0 = 15(x-2)(x+2)(x-1)(x+1)$$

CV: $x = \pm 1, \pm 2$

$$f''(x) = 15(4x^3 - 10x)$$
$$= 30x(2x^2 - 5)$$

set $f''(x) = 0$

$$0 = 30x(2x^2 - 5)$$

$x = 0$, $x = \pm\sqrt{\frac{5}{2}}$

$$2x^2 - 5 = 0$$
$$2x^2 = 5$$
$$x^2 = \frac{5}{2}$$
$$x = \pm\sqrt{\frac{5}{2}}$$

Ex. Find and classify the critical values of the function

$$f(x) = 3x^5 - 25x^3 + 60x$$

Note: Polynomials have no asymptotes, so no limits required.

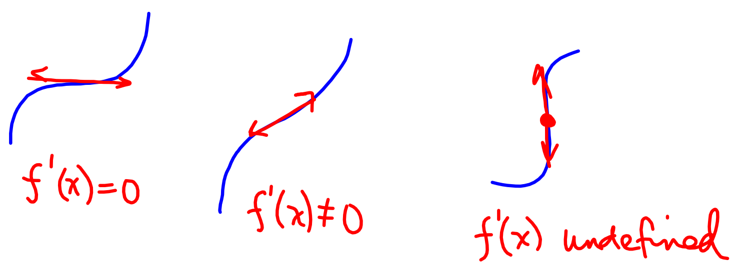
$$\begin{aligned} f'(x) &= 15(x^2-1)(x^2-4) \\ &= 15(x-1)(x+1)(x-2)(x+2) \end{aligned}$$

$$\begin{aligned} f''(x) &= 30x(2x^2-5) \\ &= 60x\left(x - \sqrt{\tfrac{5}{2}}\right)\left(x + \sqrt{\tfrac{5}{2}}\right) \end{aligned}$$

1st derivative test:

$$\begin{aligned} f'(x) &= 15(x^2-1)(x^2-4) \\ &= 15(x-1)(x+1)(x-2)(x+2) \end{aligned}$$

| interval | | -2 | | -1 | | 1 | | 2 | |
|---|---|---|---|---|---|---|---|---|---|
| $x-2$ | − | | − | | − | | − | | + |
| $x-1$ | − | | − | | − | | + | | + |
| $x+1$ | − | | − | | + | | + | | + |
| $x+2$ | − | | + | | + | | + | | + |
| $f'(x)$ | + | | − | | + | | − | | + |
| INC/DEC | INC ↗ | max | DEC ↘ | min | INC ↗ | max | DEC | min | INC |



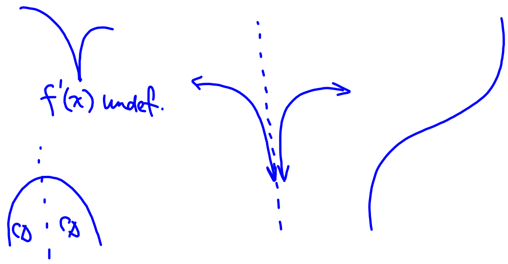

2nd derivative test:

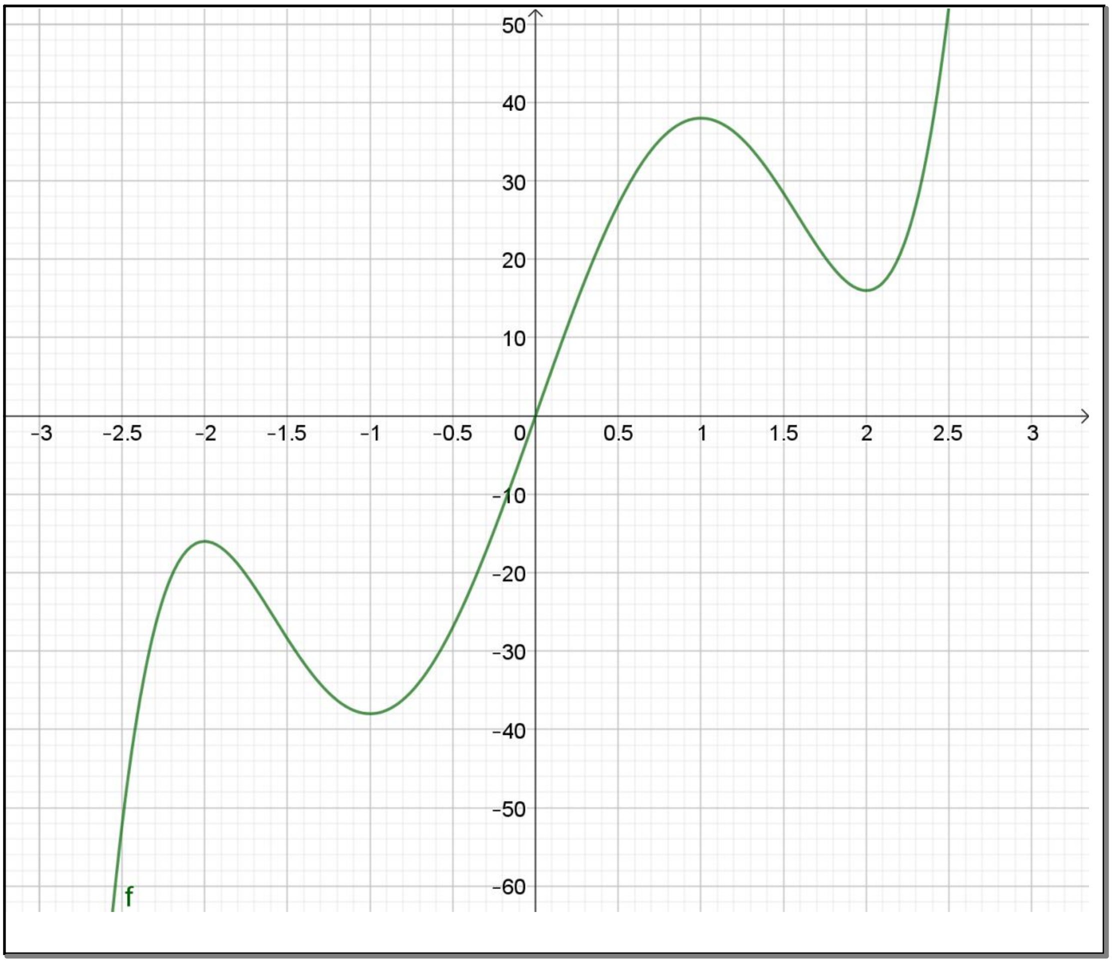
$\sqrt{\frac{5}{2}} \approx 1.58$

$$\begin{aligned} f''(x) &= 30x(2x^2-5) \\ &= 60x\left(x-\sqrt{\tfrac{5}{2}}\right)\left(x+\sqrt{\tfrac{5}{2}}\right) \end{aligned}$$

| interval | | $-\sqrt{\frac{5}{2}}=-1.6$ | | 0 | | $\sqrt{\frac{5}{2}}$ | |
|---|---|---|---|---|---|---|---|
| $x-\sqrt{\frac{5}{2}}$ | − | | − | | − | | + |
| $x$ | − | | − | | + | | + |
| $x+\sqrt{\frac{5}{2}}$ | − | | + | | + | | + |
| $f''(x)$ | − | | + | | − | | + |
| concavity | concave down CD | POI | CU | POI | CD | POI | CU |

| $x=-2$ | $x=-1$ | $x=1$ | $x=2$ |
|---|---|---|---|
| max | min | max | min |






Assigned Work:

p.205 # 2, 3, 5, 9, 10, 11, 12, 13a

b iii(b)

2(b) $y = \dfrac{25}{x^2+48} = 25(x^2+48)^{-1}$

$y' = -25(x^2+48)^{-2}(2x)$

$= \dfrac{-50x}{(x^2+48)^2}$ CV: $x=0$

$y'' = -25\left[-2(x^2+48)^{-3}(2x)(2x) + (x^2+48)^{-2}(2)\right]$

$= -25(2)(x^2+48)^{-3}\left[-4x^2 + (x^2+48)^1\right]$

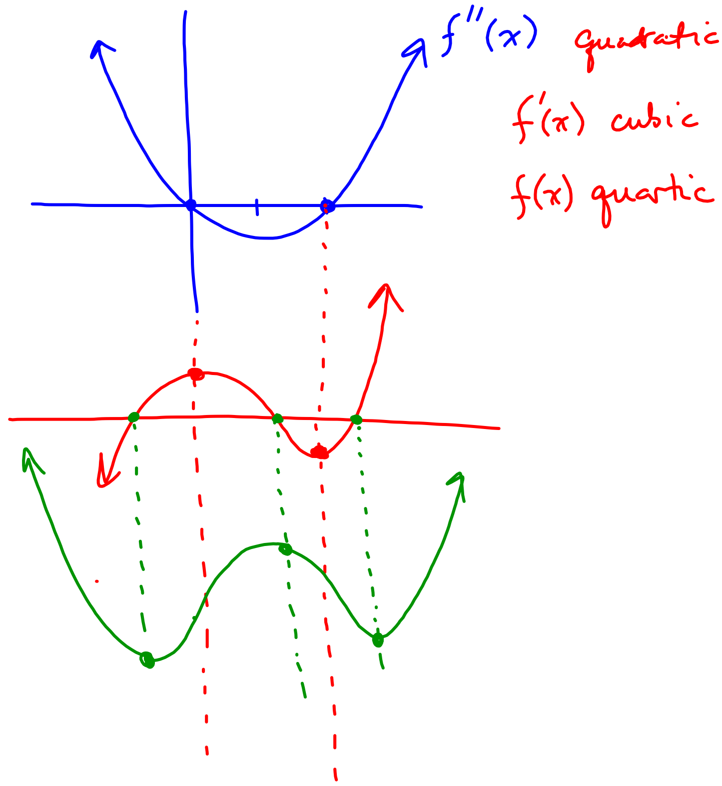
$= \dfrac{-50}{(x^2+48)^3}\left[-3x^2+48\right]$

$= \dfrac{150}{(x^2+48)^3}\left[x^2-16\right]$ CV: $x=\pm 4$

to test CV: $x=0$, sub $x=0$ into $\underbrace{y'' = \frac{150}{48^3}[-16] < 0}_{\text{Concave down at } x=0}$

$\therefore P(0, \frac{25}{48})$ is a local max.

5(2ii) b $f(0) = 2$




9. INC : $(-\infty, 2) \cup (2, 5)$

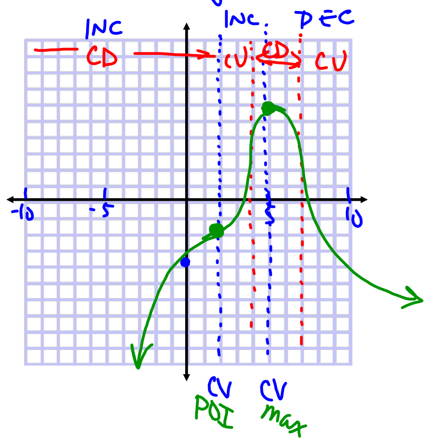
DEC : $(5, \infty)$

CV : $x = 2, 5$

CD : $(-\infty, 2) \cup (4, 7)$

CU : $(2, 4) \cup (7, \infty)$

$f(0) = -4$ y-int.



11. $f(x) = \sqrt{x+1} + \frac{b}{x}$

Solve $f''(x) = 0$

$$f(x) = (x+1)^{\frac{1}{2}} + bx^{-1}$$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} - bx^{-2}$$

$$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}} + 2bx^{-3}$$

Set $f''(x) = 0$

$$0 = \frac{-1}{4(x+1)^{3/2}} + \frac{2b}{x^3}$$

$$\frac{1}{4(x+1)^{3/2}} = \frac{2b}{x^3}$$

$$b = \frac{x^3}{8(x+1)^{3/2}} \qquad \text{sub } x = 3$$

$$b = \frac{27}{64}$$

13 $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4}$ VA: $x = \pm 2$

$$y' = \frac{(3x^2 - 4x + 4)(x^2 - 4) - (x^3 - 2x^2 + 4x)(2x)}{(x^2-4)^2}$$

$$= \frac{3x^4 - 12x^2 - \cancel{4x^3} + 16x + 4x^2 - 16 - 2x^4 + \cancel{4x^3} - 8x^2}{(x^2-4)^2}$$

$$= \frac{x^4 - 16x^2 + 16x - 16}{(x^2-4)^2}$$

OA: $x^2 - 4 \overline{\big) x^3 - 2x^2 + 4x}$ quotient $x - 2$

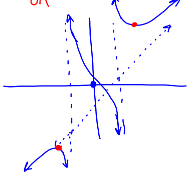
$x^3 \quad - 4x$

$-2x^2 + 8x$

$-2x^2 \quad + 8$

$8x - 8$

$$f(x) = \underbrace{x - 2}_{OA} + \frac{8x - 8}{x^2 - 4}$$



$$f(x) = \frac{x^3 - 2x^2 + 4x}{x^2 - 4}$$

$$= \frac{x(x^2 - 2x + 4)}{(x-2)(x+2)}$$

$$\lim_{x \to -2^-} f(x) = \frac{(-)(+)}{(-)(-)} = -\infty \qquad \lim_{x \to -2^+} f(x) = +\infty$$

$$\lim_{x \to 2^-} f(x) = \frac{(+)(+)}{(-)(+)} = -\infty \qquad \lim_{x \to 2^+} f(x) = \frac{(+)(+)}{(+)(+)} = +\infty$$