Derivative of Logarithmic Functions
(see p.571 - 582)

From our previous lessons on exponential functions:

$$f(x) = e^x$$
$$f'(x) = e^x$$

$$f(x) = e^{g(x)}$$
$$f'(x) = e^{g(x)}g'(x)$$

$$f(x) = b^x$$
$$f'(x) = \ln(b)b^x$$

$$f(x) = b^{g(x)}$$
$$f'(x) = \ln(\mathrm{b})\,\mathrm{b}^{g(x)}\,g'(x)$$

$\underbrace{0<b<1}_{\text{decay}} \cup \underbrace{b>1}_{\text{growth}}$

Derivative of Logarithmic Functions
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Apr. 6/2018

Recall: An exponential function, $f(x) = b^x$
has an inverse, $f^{-1}(x) = \log_b(x)$
(from Advanced Functions).

Similarly, the natural exponential function, $f(x) = e^x$
has an inverse which is the natural logarithm,

$$f^{-1}(x) = \log_e(x) = \ln(x)$$

Recall: Laws of Logarithms

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \qquad \log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_b(x^y) = y\log_b(x)$$

For the natural logarithm, they become:

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(x^y) = y\ln(x)$$

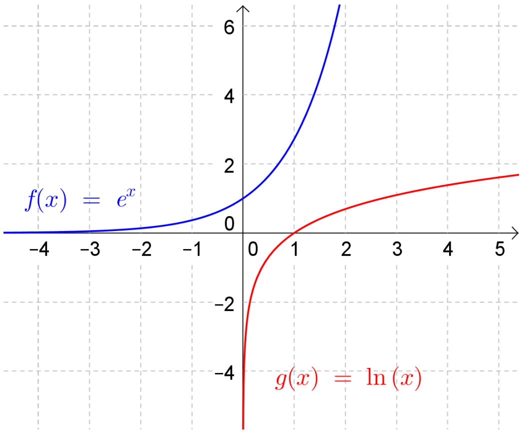
Given $f(x) = \ln x$, what is the derivative?

As we did with exponential functions, we will use graphing technology to look at the slope of the function, and look for a pattern.

see Geogebra demo: lnx Slope Demo

Given $f(x) = \ln(x)$

$$f'(x) = \frac{1}{x},\ x > 0$$



Applying the chain rule:

$$f(x) = \ln\left[g(x)\right]$$

$$f'(x) = \frac{1}{g(x)} g'(x),\ g(x) > 0$$

Ex.1 Determine the derivative for

a) $y = \ln(5x)$

b) $y = \ln(7x^2)$

c) $y = \ln\sqrt{5x}$

d) $y = 3^x \ln(x^3)$

(a) $y = \frac{1}{5x} \cdot 5$

$= \frac{1}{x}$

(b) $y' = \frac{1}{7x^2} \cdot 14x$

$= \frac{2}{x}$

(c) $y' = \frac{1}{\sqrt{5x}} \cdot \frac{1}{2}(5x)^{-\frac{1}{2}} \cdot 5$

$= \frac{5}{2} \cdot \frac{1}{5x}$

$= \frac{1}{2x}$

OR

$y = \ln(5x)^{\frac{1}{2}}$

$= \frac{1}{2}\ln(5x)$

$y' = \frac{1}{2} \cdot \frac{1}{5x} \cdot 5$

$= \frac{1}{2x}$

Ex.1 Determine the derivative for

a) $y=\ln(5x)$

b) $y=\ln(7x^2)$

c) $y=\ln\sqrt{5x}$

d) $y=\underbrace{3^x}\,\underbrace{\ln(x^3)}$

(d) $y' = (3^x \ln 3)\ln(x^3) + (3^x)\left(\frac{1}{x^3}\cdot 3x^2\right)$

$= 3^x \ln 3 \ln(x^3) + \frac{3 \cdot 3^x}{x}$

Given $f(x)=\log_b(x)$

$$f'(x)=\left(\frac{1}{x}\right)\left(\frac{1}{\ln(b)}\right)=\frac{1}{x\ln b}$$

Applying the chain rule:

$$f(x)=\log_b\left[g(x)\right]$$

$$f'(x)=\left(\frac{1}{g(x)}\right)g'(x)\left(\frac{1}{\ln(b)}\right)=\frac{g'(x)}{g(x)\ln b}$$

Ex.2 Find the derivative for

a) $y = \log_3(6x)$

b) $y = \log_{10} \sqrt[3]{x^4}$

c) $y = \sqrt{\log x}$

d) $y = \dfrac{\log_2 4x^3}{e^{5x}}$

(a) $y' = \dfrac{1}{6x \ln 3} \cdot (6)$

$= \dfrac{1}{x \ln 3}$

$x^{\frac{4}{3}}$

$\frac{4}{3} x^{\frac{1}{3}}$

(b) $y' = \dfrac{1}{\sqrt[3]{x^4} \ln 10} \cdot \frac{4}{3} x^{\frac{1}{3}}$

$= \dfrac{4x^{\frac{1}{3}}}{3x^{\frac{4}{3}} \ln 10}$

$= \dfrac{4}{3x \ln 10}$

Ex.2 Find the derivative for

a) $y = \log_3(6x)$

b) $y = \log \sqrt[3]{x^4}$

c) $y = \sqrt{\log x}$

d) $y = \dfrac{\log_2 4x^3}{e^{5x}}$

(c) $y' = \frac{1}{2}(\log x)^{-\frac{1}{2}} \cdot \dfrac{1}{x \ln 10}$

$= \dfrac{1}{2x \ln 10 \sqrt{\log x}}$

(d) $y' = \dfrac{\frac{1}{4x^3 \ln 2} \cdot (12x^2)\, e^{5x} - \log_2 4x^3 \cdot e^{5x}\,(5)}{(e^{5x})^2}$

$= \dfrac{\frac{3e^{5x}}{x \ln 2} - 5e^{5x} \log_2 4x^3}{e^{10x}}$

Assigned Work:

p.575 # 3 - 13 (Omit 5c, 9bc) 4f, 8 9a

p.578 # 1 - 5, 7, 9, 11 12, 13

p.575

4(f) $h(u) = \underbrace{e^{\sqrt{u}}}\,\underbrace{\ln\sqrt{u}}$ $\ln u^{\frac{1}{2}} = \frac{1}{2}\ln u$

$h'(u) = \frac{1}{2}e^{\sqrt{u}}\left(\frac{\ln\sqrt{u}}{\sqrt{u}} + \frac{1}{u}\right)$

$$h'(u) = e^{\sqrt{u}}\cdot\frac{1}{2}u^{-\frac{1}{2}}\ln\sqrt{u} + e^{\sqrt{u}}\cdot\frac{1}{\sqrt{u}}\cdot\frac{1}{2}u^{-\frac{1}{2}}$$
$$= \frac{e^{\sqrt{u}}\ln\sqrt{u}}{2\sqrt{u}} + \frac{e^{\sqrt{u}}}{2\sqrt{u}\sqrt{u}}$$
$$= \frac{e^{\sqrt{u}}\ln\sqrt{u}}{2\sqrt{u}} + \frac{e^{\sqrt{u}}}{2u}$$

p.575

8. $y = \ln x - 1$ // $3x - 6y - 1 = 0$

$y' = \frac{1}{x}$ $3x - 1 = 6y$

$\frac{1}{2}x - \frac{1}{6} = y$

Set y' (slope) $= \frac{1}{2}$ $m = \frac{1}{2}$

$\frac{1}{2} = \frac{1}{x}$

$x = 2$ → x-coordinate where tangent touches $y = \ln x - 1$

$y = \ln 2 - 1$

$y = mx + b$

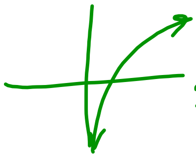
$y = \frac{1}{2}x + b$, sub $P(2, \ln 2 - 1)$

$\ln 2 - 1 = \frac{1}{2}(2) + b$

$\ln 2 - 2 = b$ $\boxed{y = \frac{1}{2}x + \ln 2 - 2}$

9 a) $f(x) = (x \ln x)^2$

where is $f'(x) = 0$



$x > 0$

$$f'(x) = 2(x \ln x)' \left[(1)\ln x + x \cdot \frac{1}{x}\right]$$

$$0 = 2\underbrace{x}\,\underbrace{\ln x}\,\underbrace{(\ln x + 1)}$$

$x = 0$

reject

$x > 0$

$\ln x = 0$

$\log_e x = 0$

$x = e^0$

$\underbrace{x = 1}_{\text{sub to find } y}$

$\ln x + 1 = 0$

$\ln x = -1$

$\log_e x = -1$

$x = e^{-1}$

$\underbrace{x = \frac{1}{e}}$

12. $\underbrace{\lim_{h \to 0} \frac{\ln(2+h) - \ln(2)}{h}}$

Same as derivative of $\ln(x)$

at $x = 2$

$f(x) = \ln(x)$

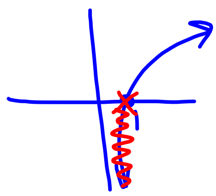
$f'(x) = \frac{1}{x}$

$f'(2) = \frac{1}{2}$

13. $f(x) = \ln(\ln x)$ $\qquad \ln(\ \ )$

(a) $f'(x) = \dfrac{1}{\ln x} \cdot \dfrac{1}{x}$

(b) inner:



$x \longrightarrow \ln x \longrightarrow \ln(\ln x) \longrightarrow y$

$\cancel{(0,\infty)} \qquad \cancel{(-\infty,\infty)}$

$(1,\infty) \longrightarrow (0,\infty) \qquad\qquad (-\infty,\infty)$

Attachments

Deriv of e^x Demo.ggb