Without using a calculator or writing anything down...

What is the SUM of the numbers from 1 to 100?

$$S = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$

101

101

101

$$50(101) = 5050$$

<u>Carl Friedrich Gauss (1777-1855)</u>

When Gauss was ten years old he was allowed to attend an arithmetic class taught by a man who had a reputation for being cynical and having little respect for the peasant children he was teaching.

The teacher had given the class a difficult summation problem in order to keep them busy and so that they might appreciate the "shortcut" formula he was preparing to teach them.

Gauss took one look at the problem, invented the shortcut formula on the spot, and immediately wrote down the correct answer.

Arithmetic and Geometric Series

June 7/2019

Definitions:

A) A series is the **sum** of the terms in a sequence. The sum is usually denoted S.

sequence: $t_1, t_2, t_3, ..., t_n$ $n \varepsilon \mathbb{N}$

series: $t_1 + t_2 + t_3 + ... + t_n = S_n$

note: $S_1 = t_1$
$S_2 = t_1 + t_2$
$S_3 = t_1 + t_2 + t_3$
etc...

Definitions:

B) An arithmetic series is the sum of the terms in an arithmetic sequence.

$$s_n = \frac{n}{2}\left[2a+(n-1)d\right]$$ or $$s_n = \frac{n}{2}\left[t_1 + t_n\right]$$

last term in sum

Recall: $t_n = a+(n-1)d$

Ex.1 Determine S_{30} for the arithmetic series

$5 + 9 + 13 + 17 + ... + (4n+1)$

t_n

$t_{30} = 4(30) + 1$

$= 121$

$S_n = \frac{n}{2}[t_1 + t_n]$

$S_{30} = \frac{30}{2}[t_1 + t_{30}]$

$= 15[5 + 121]$

$= 1890$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$a = 5 \quad d = 4$

$S_{30} = \frac{30}{2}[2(5) + (29)(4)]$

$= 15[10 + 116]$

$= 1890$

Deriving the formulae for arithmetic series:

$S_n = t_1 + t_2 + t_3 + ... + t_{n-2} + t_{n-1} + t_n$

1. Using definition $t_n = a + (n-1)d$, rewrite as

$S_n = t_1 + (t_1 + d) + (t_1 + 2d) + ... + (t_n - 2d) + (t_n - d) + t_n \quad (1)$

2. Re-arrange the order of the terms in S_n

$S_n = t_n + (t_n - d) + (t_n - 2d) + ... + (t_1 + 2d) + (t_1 + d) + t_1 \quad (2)$

3. Add equation (1) and equation (2) so that the "d's" eliminate.

$2S_n = [t_1 + t_n] + [t_1 + t_n] + [t_1 + t_n] + ... + [t_1 + t_n] + [t_1 + t_n] + [t_1 + t_n]$

4. There are n of these $[t_1 + t_n]$ terms

$2S_n = n[t_1 + t_n]$

$S_n = \frac{n}{2}[t_1 + t_n]$ This is one formula

5. Substitute $t_n = a + (n-1)d$

$S_n = \frac{n}{2}[a + a + (n-1)d]$

$S_n = \frac{n}{2}[2a + (n-1)d]$ This is the other formula

Definitions:

C) A geometric series is the sum of the terms in a geometric sequence.

$$s_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

Recall: $t_n = ar^{n-1}$

Ex.2 For the geometic series with a = 7 and r = 2, determine S_9

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{7(2^9 - 1)}{2 - 1}$$

$$= \frac{7(512 - 1)}{1}$$

$$= 3577$$

Ex.3 Determine the sum of the series

15 + 45 + 135 + 405 + ... + 32 805

t_1 t_2 . . . t_n

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$a = 15$ $r = 3$ $n = ?$

$$t_n = ar^{n-1}$$

$$32805 = 15(3)^{n-1}$$

$$2187 = 3^{n-1}$$

$$3^7 = 3^{n-1}$$

$$\Rightarrow 7 = n-1$$

$$n = 8$$

$$S_8 = \frac{15(3^8 - 1)}{3-1}$$

$$= \frac{15(6560)}{2}$$

$$= 49200$$

<u>Deriving the formula for geometric series:</u>

$$S_n = t_1 + t_2 + t_3 + ... + t_{n-2} + t_{n-1} + t_n$$

1. Rewrite using the definition $t_n = ar^{n-1}$

$$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1} \quad (1)$$

2. Multiply equation (1) by r

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \quad (2)$$

3. Subtract the two equations to eliminate most terms.

$$S_n - rS_n = a - ar^n$$

4. Factor S_n on the left side and factor a on the right side

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{(1-r)}, \ r \neq 1$$

5. To obtain the same equation as in our notes multiply both numerator and denominator by -1.

$$S_n = \frac{a(r^n - 1)}{(r-1)}, \ r \neq 1$$

Assigned Work: p.469 #2ac, 3ac, 4a, 7a, 20
p.476 #1ac, 2ac, 3ac, 4, 12a, 17

p.469 2(a)

$S_{10} = ?$ $\quad 2+4+6+\dots$

$n=10$ $\quad a=2$ $\quad +2 \; +2$ $\quad d=2$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{10}{2}\left[2(2)+(9)(2)\right]$

3(c) $100+90+80+\dots+(-50)$

$a=100$ $\quad d=-10$ $\quad t_n$ $\quad n=?$

$t_n = a+(n-1)(d)$

$-50 = 100+(n-1)(-10)$

$\frac{-150}{-10} = \frac{(n-1)(-10)}{-10}$

$15 = n-1$

$n=16$

$S_n = \frac{n}{2}\left[t_1+t_n\right]$

$S_{16} = \frac{16}{2}\left[100+(-50)\right]$

4.(a) $a=6$, $t_9=24$

$S_9 = \frac{9}{2}\left[6+24\right]$

20. $f(3)=11$ $\quad f(7)=13$ $\quad S_{20}=?$

$t_3=11$ $\quad t_7=13$ $\quad n=20$

$a=?$, $d=?$

$t_n = a+(n-1)d$

① $11 = a+2d$ $\quad\quad 13 = a+6d$ ②

$11 = a+2\left(\frac{1}{2}\right)$ $\quad\quad 11 = a+2d$

$11 = a+1$ $\quad\quad 2 = 4d$

$a=10$ $\quad\quad d=\frac{1}{2}$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{20} = \frac{20}{2}\left[2(10)+19\left(\frac{1}{2}\right)\right]$

$= 10\left[20+\frac{19}{2}\right]$

$= 200+95$

$= 295$

p. 476

3(c) $2 - 4 + 8 - \dots + 512$

t_n

$n = ?$

$a = 2 \quad r = -2$

$$t_n = ar^{n-1}$$
$$512 = 2(-2)^{n-1}$$
$$256 = (-2)^{n-1}$$
$$(-2)^8 = (-2)^{n-1}$$
$$\Rightarrow \quad 8 = n - 1$$
$$n = 9$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_9 = \frac{2((-2)^9 - 1)}{-2 - 1}$$
$$= \frac{2(-512 - 1)}{-3}$$
$$= \frac{1026}{3}$$
$$= 342$$

4. $f(1) = 0.8 = a \quad f(2) = 1.6$

$0.8 + 1.6 + \dots$

$a = 0.8 \quad r = 2$

$S_{10} \rightarrow n = 10$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{0.8(2^{10} - 1)}{2 - 1}$$
$$= \frac{0.8(1023)}{1}$$
$$=$$

17. $t_1 + t_2 + t_3 = 42$

$$\frac{t_2}{t_1} = \frac{t_3}{t_2} = r$$

$$t_3 = 3.2(t_1 + t_2)$$

$$\frac{t_3}{3.2} = t_1 + t_2$$

$$\frac{t_3}{3.2} + t_3 = 42 \quad [\times 3.2]$$

$$t_3 + 3.2t_3 = 134.4$$

$$4.2t_3 = 134.4$$

$$t_3 = 32$$