

## Trigonometric Identities

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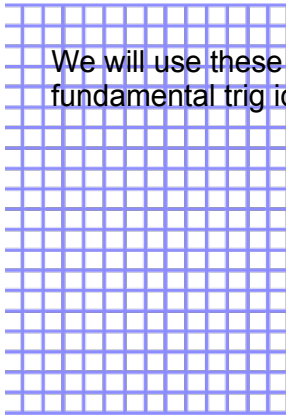
An identity is an equation which is always true for all values of the variable.

e.g.,  $(x+1)^2 = x^2 + 2x + 1$

In the x-y plane, trig ratios are expressed in terms of x, y, and r.

$$\sin \theta = \frac{y}{r} \quad \cos \theta = \frac{x}{r} \quad \tan \theta = \frac{y}{x} \quad \text{where: } r^2 = x^2 + y^2$$

We will use these definitions to develop some of the fundamental trig identities.



### 1. Quotient Identity

$$\begin{aligned} \text{Consider } \frac{\sin \theta}{\cos \theta} &= \frac{\frac{y}{r}}{\frac{x}{r}} \\ &= \frac{y}{r} \times \frac{r}{x} \\ &= \frac{y}{x} \\ \boxed{\frac{\sin \theta}{\cos \theta} = \tan \theta} \end{aligned}$$

## Conventions in Trigonometry:

- a. Brackets are required around the entire argument of a multi-term argument.

write  $\sin(x+2)$ , not  $\sin x+2$  (looks like  $\sin(x) + 2$ )  
 $\sin 25^\circ$  is acceptable, as is  $\cos x$ , or  $\tan \theta$ .

- b. Exponents (other than -1) are written between the function symbol and the argument.

$(\sin x)^2$  is written as  $\sin^2 x$

$\cos^{-1} \theta$  is the *inverse cosine*

$(\cos \theta)^{-1}$  is the reciprocal of  $\cos \theta$

$\rightarrow \frac{1}{\cos \theta}$

## 2. Pythagorean Identity

$$\text{Consider } \sin^2 \theta + \cos^2 \theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

$$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$$

$$= \frac{y^2 + x^2}{r^2}$$

recall:

$$x^2 + y^2 = r^2$$

$$= \frac{r^2}{r^2}$$

$$\boxed{\sin^2 \theta + \cos^2 \theta = 1}$$

Tips for working with trig identities:

1. Start with the most complicated side and try to make it simpler.
2. Express tangent in terms of sine and cosine.
3. Look for fundamental identities (quotient, Pythagorean).
4. Only work on one side at a time. Only switch sides if you cannot progress any further.

Ex.1 Prove  $\tan \theta \cos \theta = \sin \theta$

$$LS = \tan \theta \cos \theta \quad RS = \sin \theta$$

$$LS = \frac{\sin \theta}{\cancel{\cos \theta}} \cdot \cancel{\cos \theta}$$

$$LS = \sin \theta$$

$$LS = RS \checkmark$$

Ex.2 Prove  $\tan^2 \theta = \sin^2 \theta \cos^{-2} \theta$

$$LS = \tan^2 \theta$$

$$RS = \sin^2 \theta \cos^{-2} \theta$$

$$= \sin^2 \theta \cdot \frac{1}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$= \left( \frac{\sin \theta}{\cos \theta} \right)^2$$

$$= \tan^2 \theta$$

$$LS = RS \checkmark$$

Ex.2 Prove  $\tan^2 \theta = \sin^2 \theta \cos^{-2} \theta$

$$LS = \tan^2 \theta$$

$$RS = \sin^2 \theta \cos^{-2} \theta$$

$$= \left( \frac{\sin \theta}{\cos \theta} \right)^2$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta}{1} \cdot \frac{1}{\cos^2 \theta}$$

$$= \sin^2 \theta \cos^{-2} \theta$$

$$LS = RS$$

Ex.3 Prove  $\frac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$

$$LS = \frac{\sin^2 \theta}{1 - \cos \theta}$$

$$RS = 1 + \cos \theta$$

recall:  $\sin^2 \theta + \cos^2 \theta = 1$

$$1^2 - (\cos \theta)^2 \quad \sin^2 \theta = 1 - \cos^2 \theta$$

$$LS = \frac{1 - \cos^2 \theta}{1 - \cos \theta}$$

recall: difference of squares

$$LS = \frac{\cancel{(1 - \cos \theta)}(1 + \cos \theta)}{\cancel{(1 - \cos \theta)}}$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$LS = 1 + \cos \theta$$

$$LS = RS \checkmark$$

Assigned Work:

p.398 # 2odd, 4odd, 7odd, 10

k i c  
e e a  
i

$$2(e) \quad 1 + \frac{1}{\tan^2 x} = \frac{1}{\sin^2 x}$$

$$\begin{aligned} LS &= 1 + \frac{1}{\frac{\sin^2 x}{\cos^2 x}} \rightarrow \tan x = \frac{\sin x}{\cos x} \\ &= 1 + \frac{\cos^2 x}{\sin^2 x} \quad \tan^2 x = \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x + \cos^2 x}{\sin^2 x} \rightarrow \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{\sin^2 x} \end{aligned}$$

$$2(i) \quad \frac{1}{1 - \sin^2 x} = 1 + \tan^2 x$$

$$\begin{aligned} RS &= 1 + \tan^2 x \\ &= 1 + \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \rightarrow \sin^2 x + \cos^2 x = 1 \\ &= \frac{1}{\cos^2 x} \quad \cos^2 x = 1 - \sin^2 x \\ &= \frac{1}{1 - \sin^2 x} \end{aligned}$$

$$2(k) \quad \underline{\sin^2 x + \cos^2 x} + \tan^2 x = \frac{1}{\cos^2 x}$$

$$LS = 1 + \tan^2 x$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x} \rightarrow \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$4(c) \quad (1 - \cos^2 x) \left( 1 + \frac{1}{\tan^2 x} \right) = 1$$

$$LS = (\sin^2 x) \left( 1 + \frac{1}{\frac{\sin^2 x}{\cos^2 x}} \right)$$

$$LS = (\sin^2 x) \left( 1 + \frac{\cos^2 x}{\sin^2 x} \right)$$

$$= (\sin^2 x) \left( \frac{\sin^2 x + \cos^2 x}{\sin^2 x} \right)$$

$$= \sin^2 x \left( \frac{1}{\sin^2 x} \right)$$

$$= 1$$

$$4(i) \quad (1 + \tan^2 x)(1 - \cos^2 x) = \tan^2 x$$

$$LS = \left(1 + \frac{\sin^2 x}{\cos^2 x}\right)(\sin^2 x)$$

$$= \left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right)(\sin^2 x)$$

$$= \left(\frac{1}{\cos^2 x}\right)(\sin^2 x)$$

$$= \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x$$

$$7(a) \quad \sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$\overset{a^4}{(a^2)^2} - \overset{b^4}{(b^2)^2} = (a^2 - b^2)(a^2 + b^2)$$

$$LS = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$= (\sin^2 x - \cos^2 x)(1) \text{ Pythag. Identity}$$



$$7(c) \frac{4}{\cos^2 x} - 5 = 4 \tan^2 x - 1$$

$$RS = 4 \tan^2 x - 1$$

$$= 4 \left( \frac{\sin^2 x}{\cos^2 x} \right) - 1$$

$$= \frac{4 \sin^2 x}{\cos^2 x} - 1$$

$$= \frac{4(1 - \cos^2 x)}{\cos^2 x} - 1$$

$$= \frac{4 - 4 \cos^2 x}{\cos^2 x} - 1$$

$$= \frac{4}{\cos^2 x} - \frac{4 \cos^2 x}{\cos^2 x} - 1$$

$$= \frac{4}{\cos^2 x} - 4 - 1$$

$$= \frac{4}{\cos^2 x} - 5$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3}$$