Annuities

June 11/2012

An <u>annuity</u> is an investment where equal payments (or withdrawls) are made at regular intervals. In an <u>ordinary annuity</u> the payments/withdrawls are made at the end of each interval.

You make payments if you are saving up or paying back a loan and you make withdrawls if you have money saved up and are taking out in equal portions.

The <u>future value</u> of an annuity is the amount of money you will have (saved up) in the **future**; your deposits plus the interest each of them has earned.

The <u>present value</u> of an annuity represents the amount of money needed to invest **today** in order to provide regular payments/withdrawls over a future period of time.

Consider an investment of $700 at 4.5%/annum, compounded annually. How much would this be worth:

(a) after 0 years? 700 FV_0

(b) after 1 year? $700(1+0.045)$ FV_1

(c) after 2 years? $700(1+0.045)^2$ FV_2

(d) after 3 years? $700(1+0.045)^3$ FV_3

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$+$

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$+$

(c) after 2 years? $700(1+0.045)^2$

$+$

(d) after 3 years? $700(1+0.045)^3$

What if you had $700 available to invest at the end of each year? How much would you have after 3 years?

$FV = PV(1+r)^n$ $\quad S_n = \dfrac{a(r^n - 1)}{r-1}$ $\quad r \neq 1$

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What if you had $700 available to invest at the end of each year? How much would you have after 3 years?

By applying some common sense:

$$700 + 700(1+0.045) + 700(1+0.045)^2 + 700(1+0.045)^3$$

Consider an investment of \$700 at 4.5%/annnum, compounded annually. How much would this be worth:

(a) after 0 years? $700 \qquad = a$

(b) after 1 year? $700(1+0.045) \qquad = ar = a(1+i)$

(c) after 2 years? $700(1+0.045)^2 \qquad = ar^2 = a(1+i)^2$

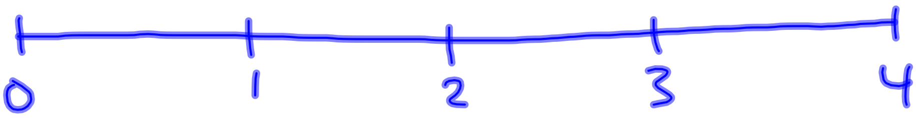
(d) after 3 years? $700(1+0.045)^3 \qquad = ar^3 = a(1+i)^3$

(e) after n years? $700(1+0.045)^n \qquad = ar^n = a(1+i)^n$

Recall: $S_n = \dfrac{a(r^n - 1)}{r-1}$ so $A = \dfrac{700\left[(1+0.045)^n - 1\right]}{0.045}$

Ex.1 Consider a four year annuity, with 12%/a interest compounded annually, for which you make annual payments of \$250.

a) Draw a time line



$250(1+0.12)^2 \quad 250(1+0.12) \quad 250$

$250(1+0.12)^3$

$250(1+0.12)^4$

* 5 payments

$\Rightarrow n=5$

Ex.1 Consider a four year annuity, with 12%/a interest compounded annually, for which you make annual payments of $250.

b) Determine the future value of each payment.

$FV_0 \doteq 393.38$ $\quad FV_3 \doteq 280.00$

$FV_1 \doteq 351.23$ $\quad FV_4 = 250$

$FV_2 \doteq 313.60$

c) Determine the future value of the annuity.

$FV = FV_0 + \dots + FV_4$

$= 1588.21$

our investment: $5 \times 250 = 1250$

interest: $1588.21 - 1250 = 338.21$

Future value formula:

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$

where *R* is the regular payment

i is the interest rate (per compounding period)

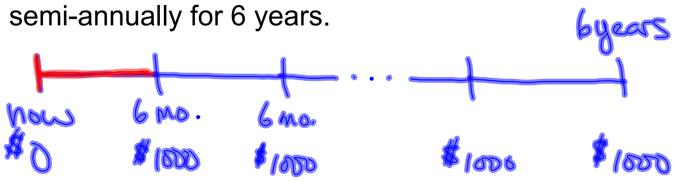
n is the number of compounding periods

(or the number of payments)

number of terms in series

Future value formula: $$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$

Ex.2 Determine the value of an annuity where $1000 is deposited at the end of each 6 months at 8%/a compounded semi-annually for 6 years.



$n = 12$ $\quad i = \frac{0.08}{2}$ ← yearly rate ← compound 2x per year

$R = 1000$ (regular payment)

$$FV = \frac{\overset{R}{1000}\left[\left(1+\overset{i}{\frac{0.08}{2}}\right)^{\overset{n}{12}} - 1\right]}{\frac{0.08}{2}}$$

$$= \frac{1000\left[\left(1+\overset{i}{0.04}\right)^{12} - 1\right]}{0.04}$$

$$\doteq 15025.81$$

Suppose you wanted to have $1000 available 3 years from now. How much should you invest at 3%/a, compounded annually?

$$PV = \frac{FV}{(1+i)^n} \quad \text{or} \quad PV = FV(1+i)^{-n}$$

$$PV = 1000(1+0.03)^{-3}$$
$$= 915.14$$

What about two years? $PV = 1000(1+0.03)^{-2}$

One year? $PV = 1000(1+0.03)^{-1}$

What is you wanted to withdraw this amount <u>each year</u> for three years?

$$1000(1+0.03)^{-1} + 1000(1+0.03)^{-2} + 1000(1+0.03)^{-3}$$

Present value formula:

$$PV = \frac{R\left[1-(1+i)^{-n}\right]}{i}$$

* now R is the regular withdrawal

4x per year

Ex.3 Determine the present value required for $75 quarterly withdrawals for 10 years at 9.6%/a compounded quarterly.

$R = 75$ $\quad n = 10 \times 4 = 40$ $\quad i = \frac{0.096}{4} = 0.024$

$$PV = \frac{75\left[1-(1+0.024)^{-40}\right]}{0.024}$$

$\doteq 1914.82$

40 payments $\times 75 = 3000$

Homework:

p.531 #7, 8, 9, 10
p.541 #8, 9, 10

p.532 #7

(a) $R = 200$ $n = 8$ $i = \frac{0.029}{12}$

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$

$$= \frac{200\left[\left(1+\frac{0.029}{12}\right)^8 - 1\right]}{\frac{0.029}{12}}$$

(b) $R = 400$

532

9. $n=4$ $\quad i=\dfrac{0.0375}{2}$ $\quad R=150$

$$FV=\frac{R\left[(1+i)^n-1\right]}{i}$$

July Jan July Jan July Jan

$0 \to 150 \longrightarrow 150(1+i)^3$

now

$150 \longrightarrow 150(1+i)^2$

$150 \to 150(1+i)$

150

$$FV=\frac{150\left[\left(1+\frac{0.0375}{2}\right)^4-1\right]}{\frac{0.0375}{2}}$$

$FV \doteq 617.0869$

10. $FV=90000$ $\quad$ 3yrs $\to 3\times 12$

$n=36$

9%/a

$\to i=\dfrac{0.09}{12}$ $\quad R=?$

$$FV=\frac{R\left[(1+i)^n-1\right]}{i}$$

$$FV\cdot i = R\left[(1+i)^n-1\right]$$

$$R=\frac{FV\cdot i}{(1+i)^n-1}$$

$$=\frac{90000\left(\frac{0.09}{12}\right)}{\left(1+\frac{0.09}{12}\right)^{36}-1}$$

$\doteq 2186.98$