There are many examples of periodic behaviour in nature. Familiar examples include the rising and setting of the sun, and the rise and fall of tides. The rhythm of the human heartbeat also follows a periodic pattern. Less obvious examples include the motion of sound waves and light waves. Even the populations of some animal species show a periodic pattern in the way they increase and decrease over time.

**Web Connection**

[www.school.mcgrawhill.ca/resources/](http://www.school.mcgrawhill.ca/resources/)

To investigate the world’s largest Ferris wheel, visit the above web site. Go to Math Resources, then to MATHEMATICS 11, to find out where to go next. Compare the largest Ferris wheel with Cosmoclock 21.

One of the world's largest Ferris wheels, Cosmoclock 21, is located in Yokohama City, Japan. The wheel has a diameter of about 100 m.

Suppose that you and a group of friends are riding the Ferris wheel. As shown in the diagram, you are at point A(50, 0) when the last of the 60 gondolas is loaded at point D. The ride then begins with you at point A. The Ferris wheel turns counterclockwise at a constant speed. The wheel takes one minute to complete one revolution.

1. Identify the coordinates of the points B, C, and D.

2. Copy and complete the following table by giving your height relative to the \(x\)-axis after the given rotations.

<table>
<thead>
<tr>
<th>Rotation of Wheel (degrees)</th>
<th>0</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
<th>450°</th>
<th>540°</th>
<th>720°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, relative to (x)-axis (metres)</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Plot the points on a grid like the one shown. Join the points with a smooth curve to show the relationship between your height, \(h\), relative to the \(x\)-axis, and the angle of rotation, \(r\).

4. **a)** Does the graph appear to be linear, quadratic, or neither?  
**b)** Describe the graph.
5. a) Predict the graph of your height, relative to the $x$-axis, versus the time, in seconds, for two complete revolutions of the wheel. Justify your reasoning.
b) Complete a table of values like the one in question 2, but replace the rotation of the wheel, in degrees, with the time, in seconds.
c) Draw the graph of your height, relative to the $x$-axis, versus time.
d) Compare your graph with your prediction from part a). Explain any differences.

A function is **periodic** if it has a pattern of $y$-values that repeats at regular intervals. One complete pattern is called a **cycle**. A cycle may begin at any point on the graph. The horizontal length of one cycle is called the **period** of the function. The period of the function in the graph shown is 4 units.

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**Example 1 Functions**

Determine whether each function is periodic. If it is, state the period.

- **a)** The graph shows a pattern of similar curves, but the pattern of $y$-values in one section of the graph does not repeat in the other sections of the graph. So, the function is not periodic.

- **b)** The pattern of $y$-values in one section of the graph repeats at regular intervals in other sections of the graph. This function is periodic.
To determine the period of this function, identify the coordinates of the point at the beginning of one cycle. Next, identify the coordinates of the point at the beginning of the next cycle.

The coordinates of the two points are (0, 4) and (8, 4). Subtract the x-coordinates.

8 - 0 = 8

The cycle repeats every 8 units, so the period of this function is 8.

**Example 2  Finding Function Values**

The graph has a period of 7. Find the value of

a) \( f(6) \)  
b) \( f(20) \)

**Solution**

a) From the graph, \( f(6) = -1 \).

b) Since the period of this function is 7, then

\[
\begin{align*}
f(6) &= f(6 + 7) \\
&= f(6 + 7 + 7) \\
&= f(20)
\end{align*}
\]

Therefore, \( f(20) = -1 \).

In general, a function \( f \) is periodic if there exists a positive number \( p \) such that \( f(x + p) = f(x) \) for every \( x \) in the domain of \( f \). The smallest positive value of \( p \) is the period of the function.

In any periodic function, the amplitude of the function is defined as half the difference between the maximum value of the function and the minimum value of the function.

For the function shown, the maximum value is 3 and the minimum value is \(-1\).

Amplitude = \( \frac{1}{2}(3 - (-1)) \)

Note that the amplitude is always positive.

\[
\begin{align*}
&= \frac{1}{2}(4) \\
&= 2
\end{align*}
\]

The amplitude of this function is 2.
**Example 3 Jogging**

André is jogging on a straight boardwalk beside the ocean. The boardwalk is 800 m long, and André begins jogging from one end at a steady pace of 8 km/h.

**a)** Graph André’s distance, in metres, from his starting point versus the time, in minutes, for four lengths of the boardwalk.

**b)** Determine the period and amplitude of the function.

**c)** State the domain and range.

**Solution**

**a)**

\[
\text{speed} = \frac{\text{distance}}{\text{time}}
\]

so, \(\text{time} = \frac{\text{distance}}{\text{speed}}\)

The boardwalk is 800 m, or 0.8 km, long.

André jogs one length of the boardwalk in \(\frac{0.8}{8}\), or 0.1 h, which is 6 min.

After 6 min, André is 800 m from his starting point. After another 6 min, for a total of 12 min, he is back at his starting point.

Therefore, three points on the graph of distance versus time are (0, 0), (6, 800), and (12, 0).

These points are joined with straight segments, because André jogs at a steady pace.

The pattern continues until André completes four lengths of the boardwalk.

The graph is as shown.

**b)**

The period of the function is 12 min.

The amplitude of the function is given by \(\frac{1}{2}(800 - 0) = 400\).

So, the amplitude is 400 m.

**c)**

The domain is \(0 \leq t \leq 24\).

The range is \(0 \leq d \leq 800\).
Key Concepts

- A function is periodic if it has a pattern of $y$-values that repeats at regular intervals.
- One complete pattern of a periodic function is called a cycle.
- The horizontal distance from the beginning of one cycle to the beginning of the next cycle is called the period.
- A function $f$ is periodic if there is a positive number $p$ such that $f(x + p) = f(x)$ for every $x$ in the domain of $f$. The length of the period is the smallest positive value of $p$.
- The amplitude of a periodic function is half the difference between the maximum value of the function and the minimum value of the function. The amplitude is always positive.

Communicate Your Understanding

The function shown is periodic. Answer questions 1–3 for this function.

1. Describe how you would determine the period of the function.
2. Describe how you would find a) $f(4)$  b) $f(5)$  c) $f(8)$  d) $f(13)$
3. Describe how you would determine the amplitude of the function.

Practise

A

1. Classify the following graphs as periodic or not periodic. Justify your decisions.
   a)
2. Determine the period and amplitude of each of the following functions.

a)

\[ y = \sin(x) \]

b)

\[ y = 2\sin(x) \]

c)

\[ y = \sin(2x) \]

d)

\[ y = \sin(\pi x) \]

3. **Communication** Sketch a graph of a periodic function with the given period and amplitude. Compare your graphs with a classmate's.

   a) a period of 6 and an amplitude of 4
   b) a period of 3 and an amplitude of 5
   c) a period of 2 and an amplitude of 2

4. A periodic function \( f \) has a period of 12. If \( f(7) = -2 \) and \( f(11) = 9 \), determine the value of

   a) \( f(43) \)  
   b) \( f(79) \)  
   c) \( f(95) \)  
   d) \( f(-1) \)

5. Determine the maximum value, the minimum value, the amplitude, the period, the domain, and the range of each function.

   a)

\[ y = \sin(x) \]

   b)

\[ y = \cos(x) \]
Apply, Solve, Communicate

6. **Application**  Anh is training by swimming lengths of a 50-m pool at a constant speed. She swims one length every minute.
   a) Graph her distance, in metres, from her starting point versus the time, in minutes, for six lengths of the pool.
   b) Determine the period and amplitude of the function.
   c) State the domain and range.
   d) After a month of training, Anh increases her speed by 10%. If she swims six lengths of the pool, how do the period and amplitude compare with the values from part b)? Explain.

7. **Inquiry/Problem Solving** Is it possible to model each situation described below with a periodic function? Explain.
   a) the sunrise times for Lively, Ontario, recorded daily for three years
   b) the number of passenger cars purchased annually in Ontario from 1990 to 2000
   c) the average monthly temperature in your community, recorded each month for two years

8. **Ferris wheel** At a county fair, the Ferris wheel has a diameter of 32 m, and its centre is 18 m above the ground. The wheel completes one revolution every 30 s.
   a) Graph a rider’s height above the ground, in metres, versus the time, in seconds, during a 2-min ride. The rider begins at the lowest position on the wheel.
   b) Determine the period and amplitude of this function.
   c) State the domain and range of the function.

9. **Climate** The mid-season high temperatures for Dorset, Ontario, were recorded over a three-year period. The results are shown in the table.
   a) Plot a graph of the temperature versus the date. Draw a periodic function that models the data as accurately as possible.
   b) Use the graph to estimate the approximate period and amplitude of the function.

<table>
<thead>
<tr>
<th>Season</th>
<th>Date</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winter</td>
<td>Feb. 5, 1998</td>
<td>-9</td>
</tr>
<tr>
<td>Spring</td>
<td>May 2, 1998</td>
<td>16</td>
</tr>
<tr>
<td>Summer</td>
<td>Aug. 3, 1998</td>
<td>25</td>
</tr>
<tr>
<td>Fall</td>
<td>Nov. 2, 1998</td>
<td>3</td>
</tr>
<tr>
<td>Winter</td>
<td>Feb. 5, 1999</td>
<td>-10</td>
</tr>
<tr>
<td>Spring</td>
<td>May 2, 1999</td>
<td>17</td>
</tr>
<tr>
<td>Summer</td>
<td>Aug. 3, 1999</td>
<td>27</td>
</tr>
<tr>
<td>Fall</td>
<td>Nov. 2, 1999</td>
<td>3</td>
</tr>
<tr>
<td>Winter</td>
<td>Feb. 5, 2000</td>
<td>-10</td>
</tr>
<tr>
<td>Spring</td>
<td>May 2, 2000</td>
<td>16</td>
</tr>
<tr>
<td>Summer</td>
<td>Aug. 3, 2000</td>
<td>26</td>
</tr>
<tr>
<td>Fall</td>
<td>Nov. 2, 2000</td>
<td>3</td>
</tr>
</tbody>
</table>
10. **Ocean cycles**  The cycle of ocean tides represents periodic behaviour and can be modelled with a periodic function. Each day, at various locations around the world, the height of the tide above the mean low-water level is recorded. Data for low tide and high tide from one location are shown in the table. Estimate the period and amplitude of the function that you would use to model the tide cycle at this location.

<table>
<thead>
<tr>
<th>Time</th>
<th>Tide Height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10:40</td>
<td>0.3</td>
</tr>
<tr>
<td>16:52</td>
<td>2.7</td>
</tr>
<tr>
<td>23:05</td>
<td>0.3</td>
</tr>
<tr>
<td>05:17</td>
<td>2.7</td>
</tr>
</tbody>
</table>

11. **Research**  Use your research skills to describe one example of a periodic function not mentioned in this section. Sketch the function, and describe its period and amplitude.

12. **Pose and solve problems**  Pose a problem related to each of the following. Check that you are able to solve each problem. Then, have a classmate solve it.

   a) ocean cycles
   b) Ferris wheels

13. If a periodic function has a whole number of cycles, what is the relationship between the period and the domain? Explain.

14. How is the amplitude of a periodic function related to its range? Explain.

15. **Earth's rotation**  A point on the Earth's equator rotates about the Earth's axis. The distance of the point from its starting position is a periodic function of the time.

   a) What is the period of the function? Explain.
   b) What is the amplitude of the function? Explain.

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**NUMBER Power**

If $\frac{9^{28} - 9^{27}}{8} = 3^x$, what is the value of $x$?