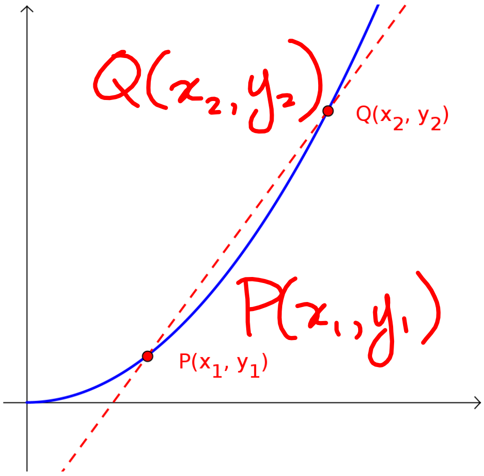
Rates of Change

Given the graph of a function, the average rate of change is defined as the slope of the secant line between two points.



$$\begin{aligned} \text{avg RoC} &= m_{PQ} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$

Using function notation, the points can also be written:

$$P(x_1, f(x_1))$$
$$Q(x_2, f(x_2))$$

$$\therefore \text{avg RoC} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

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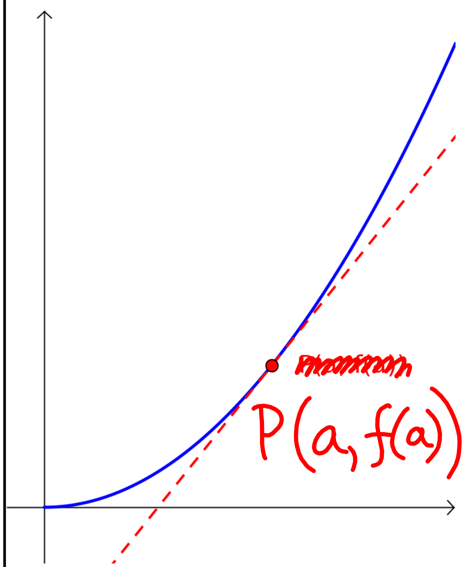
The instantaneous rate of change is the slope of the tangent line at a particular point of interest, defined by a specified value of the independent variable (e.g., at x = a).



For now, we can only estimate this value by determining the average rate of change over a very small interval near x = a.

(a) a preceding interval uses a point before the point of interest.

(b) a following interval uses a point after the point of interest.

(c) a centred interval uses points on either side of the point of interest.

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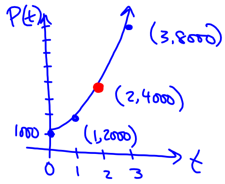
Ex.1 A bacterial colony starts with 1000 bacteria and doubles each hour.
(a) Estimate the growth rate (bacteria/hour) after 2 hours using 1 hour intervals
(i) preceding
(ii) following
(iii) centred
(b) improve the estimate using 0.1 hour intervals

$P(t) = P_0 \, 2^{t/D}$ D is doubling period

$* \; P(t) = 1000(2^t)$ D = 1 hour



(i) preceding
use (2,4000) PoI
(1,2000)

$\text{avg RoC} = \frac{4000-2000}{2-1} = \frac{2000}{1} = 2000$

∴ at 2 hours, bacteria increasing by 2000 bacteria/hour.

(ii) following
use (2,4000)
(3,8000)

$\text{avg RoC} = \frac{8000-4000}{3-2} = 4000$

(iii) centred
use (1,2000)
(3,8000)

$\text{avg RoC} = \frac{8000-2000}{3-1} = 3000$

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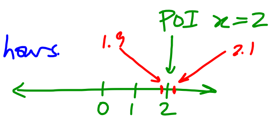
(b) interval of 0.1 hours



(i) preceding
use (2,4000)
(1.9,3732)
P(1.9)

$\text{avg RoC} \doteq \frac{4000-3732}{2-1.9} \doteq 2680$

(ii) following
(2,4000)
(2.1,4287)

$\text{avg RoC} \doteq \frac{4287-4000}{2.1-2} \doteq 2870$

(iii) centred use (1.9,3732), (2.1,4287)

$\text{avg RoC} \doteq \frac{4287-3732}{2.1-1.9} \doteq \frac{555}{0.2} \doteq 2775$

$\text{estimate} \doteq \frac{2680+2870+2775}{3} \doteq 2775$

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In general, we algebraically represent the estimated instantaneous rate of change as a difference quotient.

For $x = a$, the point of interest is $P(a, f(a))$

The following point occurs at $x = a + h$, where h is an arbitrarily small value, giving a second point

no rule, we choose what is small enough

$Q(a + h, f(a + h))$

$$\begin{aligned} \text{avg RoC} &= m_{PQ} \\ &= \frac{f(a+h)-f(a)}{(a+h)-a} \\ &= \frac{f(a+h)-f(a)}{h} \end{aligned}$$

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To estimate instantaneous rate of change:

(a) Use a series of preceding and following intervals, keeping the point of interest constant. As the intervals get smaller and smaller, look for the trend in values.

(b) Use a series of centred intervals and look for the trend.

(c) Use the difference quotient for very small values of h (both positive and negative work).

The best estimates come from the smallest intervals.

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Assigned Work:

p.76 # 8, 9
p.85 # 4, 7 9, 10, 15 (5)

4, 5
7(d)

p.85 #4 $f(x) = 6x^2 - 4$

(a) $x = -2$ -2.1 -1.9 -2 -1 0

preceding: $(-2, 20)$ $(-2.1, 22.46)$

$f(-2) = 6(-2)^2 - 4$
$= 20$

$f(-2.1) = 6(-2.1)^2 - 4$
$= 22.46$

$$\text{avg RoC} = \frac{22.46 - 20}{-2.1 - (-2)} = \frac{2.46}{-0.1} = -24.6$$

more precise, set $h = 0.01$

difference quotient (same as following)

$$\text{avg RoC} = \frac{f(a+h) - f(a)}{h} = \frac{f(-2+0.01) - f(-2)}{0.01} = \frac{f(-1.99) - f(-2)}{0.01} \doteq \frac{19.76 - 20}{0.01} \doteq -24$$

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9. $h(t) = 10 + 2t - 4.9t^2$

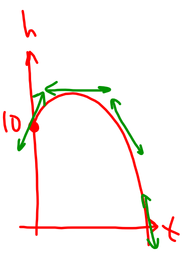
(a) set $h(t) = 0$

$0 = 10 + 2t - 4.9t^2$

$\vdots$

$t \doteq 1.65$ or $\underbrace{t \doteq -1.24}_{\text{discard, } t \geq 0}$

(b)



avg RoC at $t = 1.65$

① choose an interval size (e.g. 0.01) and average each type of RoC

② choose a series of intervals (0.1, 0.01, 0.001) and look for trend in one type of RoC

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10. $V(r) = \frac{4}{3}\pi r^3$

avg RoC @ $r = 5$ cm

start with $h = 0.1$

use $(4.9, V(4.9))$
$(5.1, V(5.1))$

avg RoC $\doteq \frac{555.65 - 492.81}{5.1 - 4.9}$

$\doteq 314.2$

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use $(4.99, V(4.99))$
$(5.01, V(5.01))$

avg RoC $\doteq \frac{V(5.01) - V(4.99)}{0.02}$

$\doteq 314.16$

use $(4.999, V(4.999))$
$(5.001, V(5.001))$

avg RoC $\doteq 314.16$

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5. $h(x) = -5x^2 + 3x + 65$

PoI at $x = 3$

try $x = 3.1$ (following)

avg RoC $= \frac{f(3.1) - f(3)}{3.1 - 3}$

$\doteq \frac{26.25 - 29}{0.1}$

$\doteq -27.5$

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try $h = 0.01$, $x = 3.01$

avg RoC $= \frac{f(3.01) - 29}{0.01}$

$\doteq \frac{28.73 - 29}{0.01}$

$\doteq -27$

avg RoC $= \frac{f(3.001) - 29}{0.001}$

$\doteq \frac{28.973 - 29}{0.001}$

$\doteq -27$

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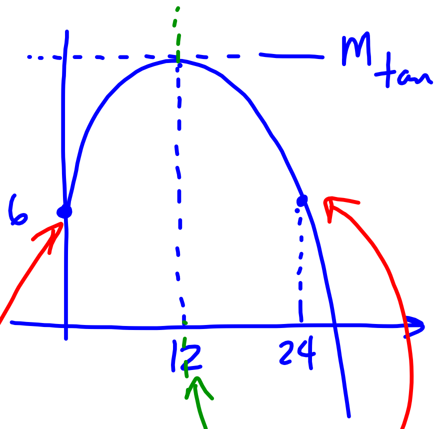
7(d)

$m_{tan} = 0$ at vertex



symmetric

vertex half-way between symmetric points

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